Lecture 18

Monday, 2 November 2026 8:40 PM

Multiwinner Voting.

Now instead of a single candidate, mechanism has to output a set of size k (k is fixed).

2 broad applications:
- select a set of "best" candidates
  - eg a badminton player to represent the county in the olympics,
    k papers in a conference
    etc.
- select a set of "representative" candidates.
  - eg seats in a legislature,
    board of a company, etc.

Representative Multiwinner Voting.

Approval voter: Each voter $i \in N$ submits a set $A_i \subseteq C$ of candidates she approves of.

For a committee W, its satisfaction of W is $|A_i \cap W|$

Thiele Rules: Fix weights: $1 = w_1 \geq w_2 \geq \dots \geq w_k \geq 0$.
For a winning set W, vote i's utility for W is
$u_i^w(W) = \sum_{t=1}^{|A_i \cap W|} \frac{1}{t}$

The Thiele rule w/ wt w elects a committee W that $\forall k$ & W maximizes
$$\sum_{i \in N} u_i(w) = \sum_{i \in N} \sum_{t=1}^{|A_i \cap W|} \frac{1}{t}$$

Some examples:
1. Approval Voting: $w = (1, 1, 1, \dots, 1)$
2. Chamberlin-Courant: $w = (1, 0, 0, \dots, 0)$
3. Proportional Approval voting: $w = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$

Example: 6 voter, $k = 3$, 5 candidates.
Voter 1-4 : $\{a, b, c\}$
5,6 : $\{d, e\}$

AV: $\{a, b, c\}$ CC: atleast 1 of abc, atleast one of de
PAV: 2 of abc, 1 of de.

Why is one rule better than another?
We need some axioms!

Intuitively: Each set of n/k voter is entitled to one representative candidate.
A group of $\geq n/k$ voter is called a "cohesive group".
For $\ell \geq 1$, a group of voter $S \subseteq N$ is ℓ-cohesive if:
1. $|S| \geq \ell \cdot \frac{n}{k}$ (so they are entitled to ℓ candidates),
2. $\left|\bigcap_{i \in S} A_i\right| \geq \ell$ (they agree on at least ℓ candidates)

In the example: eg 1, 2, 3, 4 are 2-cohesive,
5,6 are 1-cohesive.

A committee W satisfies:
Justified Representation (JR) if:
S is 1-cohesive $\Rightarrow \exists i \in S$ s.t. $|A_i \cap W| \geq 1$
(at least one voter is 1-satisfied)

Proportional JR if:
S is ℓ-cohesive $\Rightarrow \left|\bigcup_{i \in S} (A_i \cap W)\right| \geq \ell$
(How satisfied S is $\geq \ell$)

Extended JR if
S is ℓ-cohesive $\Rightarrow \exists i \in S$ s.t. $|A_i \cap W| \geq \ell$
(at least one voter is ℓ-satisfied)

Claim: EJR $\Rightarrow$ PJR $\Rightarrow$ JR.
(easy)

Say voting rule satisfies JR/PJR/EJR if every set of candidates output by the rule satisfies the axiom.

From ex example: AV does not satisfy JR
(from ex example)
CC does not satisfy PJR
Claim: CC satisfies JR
Proof: Suppose not, then $\exists W$ s.t. $|W| = k$, W maximizes $|\{i : A_i \cap W \neq \emptyset\}|$. W doesn't satisfy JR, so $\exists S \subseteq N$ s.t. $|S| \geq \frac{n}{k}$, $\exists \ell \in \bigcap_{i \in S} A_i$, & $\forall i \in S$, $A_i \cap W = \emptyset$.
WTS: show $w \in W$ s.t. $(W \setminus w) \cup \ell$ has higher C-C score.
Order the candidates $w_1 \dots w_k$
In this order, let $N_j = \{i \in N : w_j \in A_i,\ w_1 \dots w_{j-1} \notin A_i\}$
(N_j is the set of voter that approve w_j but not $w_1 \dots w_{j-1}$)
Then 1. N_1, N_2 are disjoint for $i \neq j$
2. $\sum_{j=1}^{k} |N_j|$ = total C-C score of W. $\leq n \cdot k$

$\Rightarrow \exists f$: $|N_f| \leq \frac{n}{k} - 1$

The replacing w_f by ℓ must increase the C-C score ☠

However:
Claim: Maximizing the C-C score is NP-hard.
(prove yourself)
However finding a committee that satisfies JR can be done in polynomial time.

Greedy: Pick candidate that cover the max # of uncovered voters.
Algo: $U \leftarrow N$, $W \leftarrow \emptyset$
For $i = 1 \dots k$
Let w maximize $|\{i \in U : w \in A_i\}|$
$W \leftarrow W \cup \{w\}$, $U \leftarrow U \setminus \{i \in N : w \in A_i\}$

Prove yourself this works.

Claim: PAV satisfies EJR.
Proof: Suppose W is a PAV-optimal committee. So W maximizes $\sum_{i \in N} \sum_{t=1}^{|A_i \cap W|} \frac{1}{t}$.

If W does not satisfy EJR, $\exists S, \ell$ s.t. $|S| \geq \ell \cdot n/k$, $\left|\bigcap_{i \in S} A_i\right| \geq \ell$. but $\forall i \in S$, $|A_i \cap W| \leq \ell - 1$

Thus $\exists c \in \bigcap_{i \in S} A_i$, $c \notin W$.
WTS: show $\exists w \in W$ s.t. $(W \setminus w) \cup c$ has higher PAV score.
Let $W' = W \cup c$.
$\forall w \in W'$, $\Delta(w) = \text{score}(W') - \text{score}(W' \setminus w) = \sum_{i : w \in A_i} \frac{1}{|W' \cap A_i|}$

Thus, $\text{score}(W \setminus w \cup c) - \text{score}(W)$
$= \text{score}(W' \setminus w) - \text{score}(W' \setminus c)$
$= \Delta(c) - \Delta(w)$

Now $\Delta(c) \geq \ell \cdot n \cdot \frac{1}{\ell} \cdot \frac{1}{k} = \frac{n}{k}$

Also $\sum_{w \in W} \Delta(w) = \sum_{w \in W} \sum_{i : w \in A_i} \frac{1}{|W' \cap A_i|} = \sum_{i} \sum_{w \in A_i \cap W} \frac{1}{|W' \cap A_i|}$
$= 1$

Thus $\exists w \in W'$ s.t. $\Delta(w) \leq \frac{n}{k+1}$